

GENERALISED ENERGY CONSERVATION LAW FOR WAVE EQUATIONS WITH VARIABLE PROPAGATION SPEED

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ABSTRACT. We investigate the long time behaviour of the L^2 -energy of solutions to wave equations with variable speed. The novelty of the approach is the combination of estimates for higher order derivatives of the coefficient with a stabilisation property.

1. MODEL PROBLEM

We consider the Cauchy problem

$$(1.1) \quad u_{tt} - a^2(t)\Delta u = 0, \quad u(0, \cdot) = u_1 \in H^1(\mathbb{R}^n), \quad D_t u(0, \cdot) = u_2 \in L^2(\mathbb{R}^n)$$

for a wave equation with variable propagation speed. As usual we denote $D_t = -i\partial_t$, $\Delta = \sum_j \partial_{x_j}^2$ the Laplacian on \mathbb{R}^n and $a^2(t)$ is a sufficiently regular non-negative function subject to conditions specified later on. We are interested in the behaviour of the energy as $t \rightarrow \infty$ for coefficients bearing *very fast oscillations* (in the classification of Reissig-Yagdjian [1], [2]), but satisfying a suitable *stabilisation condition* in the spirit of Hirose [3], [4]. For this we assume that the coefficient $a(t)$ can be written as product

$$(1.2) \quad a(t) = \lambda(t)\omega(t)$$

of a shape function $\lambda(t)$ (being essentially free of oscillations) and a bounded perturbation $\omega(t)$ containing a certain amount of oscillations controlled by our main assumptions.

Our method leads to an extension of the generalised energy conservation law from [3] including the shape function $\lambda(t)$. Roughly speaking, this means that the adapted hyperbolic energy of the solution $u(t, x)$ of (1.1),

$$(1.3) \quad \mathbb{E}_\lambda(t; u) = \frac{1}{2} \int_{\mathbb{R}^n} (\lambda^2(t) |\nabla u(t, x)|^2 + |u_t(t, x)|^2) \, dx$$

satisfies a *two-sided* energy inequality of the form

$$(1.4) \quad C_1 \leq \frac{1}{\lambda(t)} \mathbb{E}_\lambda(t; u) \leq C_2$$

with constants C_1 and C_2 depending on the data. The upper bound can be given in terms of the norms of $u_1 \in H^1(\mathbb{R}^n)$ and $u_2 \in L^2(\mathbb{R}^n)$, it is *not* possible to replace $H^1(\mathbb{R}^n)$ by the corresponding homogeneous space $\dot{H}^1(\mathbb{R}^n)$ (as in the case of [3]).

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The behaviour of the energy is only of interest as $t \rightarrow \infty$ (or in the neighbourhood of zeros of $\lambda(t)$, which is not within the scope of this note). Therefore it is reasonable to restrict considerations to *monotonous* $\lambda(t)$ with $\lambda(0) > 0$.

Basic assumptions of our approach are that $a(t) \in C^m(\mathbb{R}_+)$, $m \geq 2$, together with

(A1): $\lambda(t) > 0$, $\lambda'(t) > 0$ together with the estimates

$$(1.5) \quad \lambda'(t) \approx \lambda(t) \left(\frac{\lambda(t)}{\Lambda(t)} \right), \quad |\lambda''(t)| \lesssim \lambda(t) \left(\frac{\lambda(t)}{\Lambda(t)} \right)^2,$$

where $\Lambda(t) = 1 + \int_0^t \lambda(s) ds$ denotes a primitive of $\lambda(t)$;

(A2): $0 < c_1 \leq \omega(t) \leq c_2$;

(A3): $\omega(t)$ λ -stabilises towards 1, i.e. we assume that

$$(1.6) \quad \int_0^t \lambda(s) |\omega(s) - 1| ds \lesssim \Theta(t) \ll \Lambda(t), \quad t \rightarrow \infty;$$

(A4): for $k = 1, 2, \dots, m$ the symbol type estimates

$$(1.7) \quad \left| d_t^k a(t) \right| \lesssim \lambda(t) \Xi^{-k}(t)$$

are valid, where $\lambda(t) \Xi(t) \gtrsim \Theta(t)$ and

(A5):

$$(1.8) \quad \int_t^\infty \lambda^{1-m}(s) \Xi^{-m}(s) ds \lesssim \Theta^{1-m}(t).$$

The number m is determined from (A3)–(A5). The conditions are similar to those from [4], reason for that is the close relation between wave equations with increasing propagation speed and weakly damped ones. Condition (A5) can be understood as defining property of $\Xi(t)$ in terms of $\lambda(t)$, the stabilisation rate $\Theta(t)$ and the number m .

Stabilisation condition (A3) is only meaningful if $m \geq 2$. Indeed if (A4) and (A5) hold with $m = 1$ we would require $a'(t)/a(t) \in L^1(\mathbb{R}_+)$ and two-sided energy estimates follow directly by Gronwall inequality.

In most examples it is useful to replace assumptions (A4) and (A5) by the following two slightly stronger conditions, namely one can use a specific function $\Xi(t)$ depending on $\lambda(t)$, the stabilisation rate $\Theta(t)$ and the number m and assume that

(A4'): for $k = 1, 2, \dots, m$ the symbol type estimates

$$(1.9) \quad \left| d_t^k a(t) \right| \lesssim \lambda(t) \left(\frac{\lambda(t)}{\Theta(t)} \left(\frac{\Theta(t)}{\Lambda(t)} \right)^{\frac{1}{m}} \right)^k$$

are valid and

(A5'): for some number $\epsilon > 0$ the estimate $\Lambda^\epsilon(t) \lesssim \Theta(t)$ holds true.

The advantage is that these conditions are more easily checked and the benefit of the number m can be seen directly. Condition (A4') is satisfied for all m if

(A4'') for any $\epsilon > 0$ and all k the symbol type estimates

$$(1.10) \quad \left| d_t^k a(t) \right| \lesssim \lambda(t) \left(\frac{\lambda(t)}{\Theta(t)} \left(\frac{\Theta(t)}{\Lambda(t)} \right)^\epsilon \right)^k$$

hold true.

Later on we will construct examples along the lines of these conditions and also give counter-examples in the sense that there exists a coefficient satisfying the converse to the inequality (A4'') for $\epsilon < 0$ arbitrarily close to 0 such that the mentioned uniform estimates of the energy do not hold.

Notational remark: We use the notation $f \lesssim g$ for two positive functions if there exists a constant C such that $f \leq Cg$ for all values of the arguments. Similarly $f \gtrsim g$ if $g \lesssim f$ and $f \approx g$ if both $f \lesssim g$ and $g \lesssim f$ are true. Further we denote $f \ll g$ if the quotient is bounded away from 1, i.e. if $f/g \leq c < 1$ uniformly in all arguments. For matrices $\|\cdot\|$ denotes the spectral norm, any other matrix-norm will do as well. Additionally we use $|\cdot|$ for the matrix of the absolute values.

2. REPRESENTATION OF SOLUTIONS

We will not solve (1.1) directly, we will reformulate it as a system of first order and consider the fundamental solution to that system instead. To be more precise, we apply a partial Fourier transform to reduce (1.1) to an parameter-dependent ordinary differential equation, $\hat{u}_{tt} + a^2(t)|\xi|^2 \hat{u} = 0$, and consider as new unknown the vector

$$(2.1) \quad V(t, \xi) = (\lambda(t)|\xi| \hat{u}, D_t \hat{u})^T.$$

We include $\lambda(t)$ to resemble the energy $\mathbb{E}_\lambda(u; t) = \|V(t, \xi)\|_{L^2}^2$. We could include $a(t)$ instead, but in view of (A2) this does not change much. The vector-valued function $V(t, \xi)$ satisfies the first order system

$$(2.2) \quad D_t V = \begin{pmatrix} \frac{D_t \lambda(t)}{\lambda(t)} & \lambda(t)|\xi| \\ \lambda(t)\omega^2(t)|\xi| & \end{pmatrix} V,$$

whose coefficient matrix will be denoted as $A(t, \xi)$. Our aim is to construct the corresponding fundamental solution, i.e. the matrix-valued solution to

$$(2.3) \quad D_t \mathcal{E}(t, s, \xi) = A(t, \xi) \mathcal{E}(t, s, \xi), \quad \mathcal{E}(s, s, \xi) = I \in \mathbb{C}^{2 \times 2}.$$

If we set formally $\omega(t) = 1$ we obtain a much simpler system (by assumption (A1)). Due to its importance for our approach, we denote the corresponding coefficient matrix as $A_\lambda(t, \xi)$ and the corresponding fundamental solution as $\mathcal{E}_\lambda(t, s, \xi)$. It will be considered first and (partly) constructed in Section 2.1.

2.1. What makes $\lambda(t)$ nice? In a first step we consider the problem with a monotone coefficient. We construct $\mathcal{E}_\lambda(t, s, \xi)$ for $s, t \geq t_\xi^{(1)}$, where the zone boundary $t_\xi^{(1)}$ is given implicitly by

$$(2.4) \quad \Lambda(t_\xi^{(1)})|\xi| = N$$

for some fixed constant N (chosen to be sufficiently large) and prove the following statement:

Lemma 2.1. *Assume (A1). Then the fundamental solution $\mathcal{E}_\lambda(t, s, \xi)$ satisfies uniformly in $s, t \geq t_\xi^{(1)}$ the two-sided estimate*

$$(2.5) \quad \|\mathcal{E}_\lambda(t, s, \xi)\| \approx \frac{\sqrt{\lambda(t)}}{\sqrt{\lambda(s)}}$$

(regardless of the order of s and t).

The proof of this fact is essentially given by a C^2 -theory (in the language of [3], [4]) and follows the corresponding result from [5].

Proof. We apply two steps of transformations to the Cauchy problem $D_t V_\lambda = A_\lambda(t, \xi) V_\lambda$. In a first one we set $V_\lambda^{(0)} = M^{-1} V_\lambda$, where

$$(2.6) \quad M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad M^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

is a diagonaliser of the $|\xi|$ -homogeneous part of $A_\lambda(t, \xi)$. This yields the new system

$$(2.7) \quad D_t V_\lambda^{(0)} = \left(\begin{pmatrix} \lambda(t)|\xi| & \\ & -\lambda(t)|\xi| \end{pmatrix} + \frac{D_t \lambda(t)}{2\lambda(t)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right) V_\lambda^{(0)}.$$

For convenience we denote the first (diagonal) matrix as $\mathcal{D}_\lambda(t, \xi)$ and the second (remainder) as $R_{0,\lambda}(t, \xi)$. In a second step we want to transform the remainder, keeping the structure of the main diagonal part. For this we set

$$(2.8a) \quad N_\lambda(t, \xi) = I + \frac{D_t \lambda(t)}{4\lambda^2(t)|\xi|} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

$$(2.8b) \quad F_\lambda(t, \xi) = \frac{D_t \lambda(t)}{2\lambda(t)} I,$$

such that the commutator relation

$$(2.9) \quad [\mathcal{D}_\lambda(t, \xi), N_\lambda(t, \xi)] + R_{\lambda,0}(t, \xi) - F_\lambda(t, \xi) = 0$$

holds true. This relation implies that

$$(2.10) \quad \begin{aligned} B(t, \xi) &= (D_t - \mathcal{D}_\lambda(t, \xi) - R_{\lambda,0}(t, \xi))N_\lambda(t, \xi) - N_\lambda(t, \xi)(D_t - \mathcal{D}_\lambda(t, \xi) - F_\lambda(t, \xi)) \\ &= D_t N_\lambda(t, \xi) - [\mathcal{D}_\lambda(t, \xi), N_\lambda(t, \xi)] - R_{0,\lambda}(t, \xi)N_\lambda(t, \xi) + N_\lambda(t, \xi)F_\lambda(t, \xi) \\ &= D_t N_\lambda(t, \xi) - R_{\lambda,0}(t, \xi)(N_\lambda(t, \xi) - I) + (N_\lambda(t, \xi) - I)F_\lambda(t, \xi) \end{aligned}$$

is bounded by

$$(2.11) \quad \|B(t, \xi)\| \lesssim \left| D_t \frac{D_t \lambda(t)}{\lambda^2(t)|\xi|} \right| + \left| \frac{(D_t \lambda(t))^2}{\lambda^3(t)|\xi|} \right| \lesssim \frac{\lambda(t)}{\Lambda^2(t)|\xi|}$$

as consequence of assumption (A1). Furthermore, $\|N_\lambda(t, \xi)\| \lesssim 1 + \frac{1}{\Lambda(t)|\xi|} \lesssim 1$ combined with

$$(2.12) \quad \det N_\lambda(t, \xi) = 1 - \frac{(\partial_t \lambda(t))^2}{16\lambda^4(t)|\xi|^2} \geq 1 - \frac{C}{N}$$

implies that for sufficiently large N the matrix $N_\lambda(t, \xi)$ is invertible with uniformly bounded inverse, $\|N_\lambda^{-1}(t, \xi)\| \lesssim 1$. This fixes the choice of N for now (until we may make it slightly larger later on).

Setting $V_\lambda^{(1)} = N_\lambda^{-1}(t, \xi)V_\lambda^{(0)}$ we obtain the system

$$(2.13) \quad D_t V_\lambda^{(1)} = (\mathcal{D}_\lambda(t, \xi) + F_\lambda(t, \xi) + R_{\lambda,1}(t, \xi)) V_\lambda^{(1)}$$

with remainder $R_{\lambda,1}(t, \xi) = -N_\lambda^{-1}(t, \xi)B(t, \xi)$ satisfying the bound (2.11). This system can be solved in two steps. First consider the diagonal part $D_t - \mathcal{D}_\lambda(t, \xi) - F_\lambda(t, \xi)$. The corresponding fundamental solution is

$$(2.14) \quad \begin{aligned} \tilde{\mathcal{E}}_{\lambda,1}(t, s, \xi) &= \exp \left(\int_s^t (\mathcal{D}_\lambda(\tau, \xi) + F_\lambda(\tau, \xi)) d\tau \right) \\ &= \frac{\sqrt{\lambda(t)}}{\sqrt{\lambda(s)}} \text{diag} \left(e^{i(\Lambda(t) - \Lambda(s))|\xi|}, e^{-i(\Lambda(t) - \Lambda(s))|\xi|} \right) \end{aligned}$$

with $\text{cond } \tilde{\mathcal{E}}_{\lambda,1}(t, s, \xi) = \|\tilde{\mathcal{E}}_{\lambda,1}(t, s, \xi)\| \|\tilde{\mathcal{E}}_{\lambda,1}(s, t, \xi)\| = 1$. Now, we make the *ansatz* $\mathcal{E}_{\lambda,1}(t, s, \xi) = \tilde{\mathcal{E}}_{\lambda,1}(t, s, \xi) \mathcal{Q}_{\lambda,1}(t, s, \xi)$ for the fundamental solution to (2.13). A simple calculation yields for the unknown $\mathcal{Q}_{\lambda,1}$ the following equation

$$(2.15) \quad D_t \mathcal{Q}_{\lambda,1}(t, s, \xi) = \tilde{\mathcal{E}}_{\lambda,1}(s, t, \xi) R_{\lambda,1}(t, \xi) \tilde{\mathcal{E}}_{\lambda,1}(t, s, \xi) \mathcal{Q}_{\lambda,1}(t, s, \xi), \quad \mathcal{Q}_{\lambda,1}(s, s, \xi).$$

The matrix $\mathcal{R}_{\lambda,1}(t, s, \xi) = \tilde{\mathcal{E}}_{\lambda,1}(s, t, \xi) R_{\lambda,1}(t, \xi) \tilde{\mathcal{E}}_{\lambda,1}(t, s, \xi)$ satisfies the bound (2.11),

$$(2.16) \quad \|\mathcal{R}_{\lambda,1}(t, s, \xi)\| \lesssim \frac{\lambda(t)}{\Lambda^2(t)|\xi|},$$

such that the representation of $\mathcal{Q}_\lambda(t, s, \xi)$ by means of a Peano-Baker series

$$(2.17) \quad \mathcal{Q}_{\lambda,1}(t, s, \xi) = I + \sum_{k=1}^{\infty} \int_s^t \mathcal{R}_{\lambda,1}(t_1, s, \xi) \cdots \int_s^{t_{k-1}} \mathcal{R}_{\lambda,1}(t_k, s, \xi) dt_k \cdots dt_1$$

implies the uniform bound

$$(2.18) \quad \begin{aligned} \|\mathcal{Q}_{\lambda,1}(t, s, \xi)\| &\leq \exp \left(\int_s^t \|\mathcal{R}_{\lambda,1}(\tau, s, \xi)\| d\tau \right) \leq \exp \left(C \int_{t_\xi^{(1)}}^\infty \frac{\lambda(\tau)}{\Lambda^2(\tau)|\xi|} d\tau \right) \\ &\leq \exp \left(\frac{C}{\Lambda(t_\xi^{(1)})|\xi|} \right) \leq \exp \left(\frac{C}{N} \right) \lesssim 1. \end{aligned}$$

The representation $\mathcal{E}_\lambda(t, s, \xi) = M N_\lambda(t, \xi) \tilde{\mathcal{E}}_{\lambda,1}(t, s, \xi) \mathcal{Q}_{\lambda,1} N_\lambda^{-1}(s, \xi) M^{-1}$ of the fundamental solution together with the bounds of all factors established above gives the desired norm estimate. This completes the proof. \square

Remark 2.1. In fact we have established more than stated in Lemma 2.1. We have a precise description of the structure of the fundamental solution $\mathcal{E}_\lambda(t, s, \xi)$ which allows to

track the large time asymptotics of solutions. To be more precise, we have $N_\lambda(t, \xi) \rightarrow \mathbf{I}$ as $t \rightarrow \infty$ for fixed $\xi \neq 0$ together with $\mathcal{Q}_{\lambda,1}(t, s, \xi) \rightarrow \mathcal{Q}_{\lambda,1}(\infty, s, \xi)$, where

$$(2.19) \quad \mathcal{Q}_{\lambda,1}(\infty, s, \xi) = \mathbf{I} + \sum_{k=1}^{\infty} \int_s^{\infty} \mathcal{R}_{\lambda,1}(t_1, s, \xi) \cdots \int_s^{t_{k-1}} \mathcal{R}_{\lambda,1}(t_k, s, \xi) dt_k \cdots dt_1$$

and

$$(2.20) \quad \begin{aligned} \|\mathcal{Q}_{\lambda,1}(t, s, \xi) - \mathcal{Q}_{\lambda,1}(\infty, s, \xi)\| &\leq \exp\left(\int_t^{\infty} \|R_{\lambda,1}(\tau, \xi)\| d\tau\right) - 1 \\ &\leq \exp\left(\frac{C}{\Lambda(t)|\xi|}\right) - 1 \lesssim \frac{C}{\Lambda(t)|\xi|} \rightarrow 0, \quad t \rightarrow \infty. \end{aligned}$$

Both convergences are locally uniform in $\xi \neq 0$. Roughly speaking this means that the solutions are determined for large time by $M\tilde{\mathcal{E}}_{\lambda,1}(t, s, \xi)M^{-1}$, which is just a free wave (where $\lambda \equiv 1$) with a substitution in the time-variable.

2.2. Treatment in the pseudo-differential zone. We denote

$$(2.21) \quad Z_{pd}(N) = \{(t, \xi) : 0 \leq t \leq t_\xi^{(1)}\}$$

as pseudo-differential zone and continue the construction of the fundamental solution inside this set. For this we consider $\mathcal{E}(t, t_\xi^{(1)}, \xi)$ and represent its entries as solutions of certain Volterra-type integral equations. Thus, we solve the problem *backwards*.

Lemma 2.2. *Assume (A1) and (A2). Then uniformly in $Z_{pd}(N)$ the estimate*

$$(2.22) \quad |\mathcal{E}(t, s, \xi)| \lesssim \begin{pmatrix} \frac{\lambda(t)}{\lambda(s)} & \frac{\lambda(t)(s-t)}{\Lambda(s)} \\ \frac{\lambda(t)}{\lambda(s)} & 1 \end{pmatrix}, \quad t \leq s,$$

holds true.

Note, that Liouville theorem applied to the original system immediately gives the representation of the determinant

$$(2.23) \quad \det \mathcal{E}(t, s, \xi) = \frac{\lambda(t)}{\lambda(s)},$$

which means that we can conclude estimates for the inverse matrix by Cramer's rule.

Corollary 2.3. *Uniformly in $Z_{pd}(N)$ the fundamental solution $\mathcal{E}(t, s, \xi)$ satisfies*

$$(2.24) \quad |\mathcal{E}(t, s, \xi)| \lesssim \frac{\lambda(t)}{\lambda(s)} \begin{pmatrix} 1 & \frac{\lambda(s)(t-s)}{\Lambda(t)} \\ \frac{\lambda(s)}{\lambda(t)} & \frac{\lambda(s)}{\lambda(t)} \end{pmatrix} \lesssim \begin{pmatrix} \frac{\lambda(t)}{\lambda(s)} & \frac{\lambda(t)(t-s)}{\Lambda(t)} \\ 1 & 1 \end{pmatrix}, \quad s \leq t.$$

Proof. (of Lemma 2.2) We consider the columns of $\mathcal{E}(t, s, \xi)$ separately and rewrite the differential equation (2.3) as system of integral equations. This gives for the entries

$v(t, \xi)$ and $w(t, \xi)$ of one column

$$(2.25a) \quad v(t, \xi) = \frac{\lambda(t)}{\lambda(s)} v(s, \xi) + i|\xi| \lambda(t) \int_s^t w(\tau, \xi) d\tau,$$

$$(2.25b) \quad w(t, \xi) = w(s, \xi) + i|\xi| \lambda(t) \int_s^t \omega^2(\tau) v(\tau, \xi) d\tau$$

with appropriate data $v(s, \xi)$ and $w(s, \xi)$.

First column. We set $v(s, \xi) = 1$ and $w(s, \xi) = 0$ and restrict to the range $0 \leq t \leq s$. Plugging the second integral equation into the first yields

$$(2.26) \quad \begin{aligned} v(t, \xi) &= \frac{\lambda(t)}{\lambda(s)} - |\xi|^2 \lambda(t) \int_t^s \lambda(\tau) \int_\tau^s \omega^2(\theta) v(\theta, \xi) d\theta d\tau \\ &= \frac{\lambda(t)}{\lambda(s)} - |\xi|^2 \lambda(t) \int_t^s \left(\int_t^\theta \lambda(\tau) d\tau \right) \omega^2(\theta) v(\theta, \xi) d\theta. \end{aligned}$$

The best we can expect is an estimate of the form $\lambda(s)v(t, \xi)/\lambda(t) \in L^\infty(Z'_{pd}(N))$, where $Z'_{pd}(N) = \{(t, s, \xi) : 0 \leq t \leq s \leq t_\xi^{(1)}\}$. Rewriting the integral equation gives

$$(2.27) \quad \frac{\lambda(s)v(t, \xi)}{\lambda(t)} = 1 + \int_t^s k_1(t, \theta, \xi) \frac{\lambda(s)v(\theta, \xi)}{\lambda(\theta)} d\theta$$

with kernel

$$(2.28) \quad k_1(t, \theta, \xi) = -|\xi|^2 \omega^2(\theta) \lambda(\theta) \int_t^\theta \lambda(\tau) d\tau, \quad \theta \in [t, s].$$

Now the kernel estimate

$$(2.29) \quad \begin{aligned} \sup_{(t, \xi) \in Z_{pd}} \int_0^s \sup_{0 \leq \tilde{t} \leq \theta} |k_1(\tilde{t}, \theta, \xi)| d\theta &\lesssim |\xi|^2 \int_0^{t_\xi^{(1)}} \lambda(\theta) \int_0^\theta \lambda(\tau) d\tau d\theta \\ &= |\xi|^2 \int_0^{t_\xi^{(1)}} \Lambda(\theta) \lambda(\theta) d\theta = \frac{1}{2} |\xi|^2 \Lambda^2(t_\xi^{(1)}) \lesssim 1 \end{aligned}$$

uniform in $Z'_{pd}(N)$ implies that the Neumann series

$$(2.30) \quad \frac{\lambda(s)v(t, \xi)}{\lambda(t)} = 1 + \sum_{j=1}^{\infty} \int_t^s k_1(t, t_1, \xi) \cdots \int_{t_{k-1}}^s k_1(t, t_k, \xi) dt_k \cdots dt_1$$

converges in $L^\infty(Z'_{pd}(N))$ (for arbitrary N). Therefore, as claimed,

$$(2.31) \quad |v(t, \xi)| \lesssim \frac{\lambda(t)}{\lambda(s)},$$

and the second integral equation implies the corresponding bound for $w(t, \xi)$,

$$(2.32) \quad |w(t, \xi)| \lesssim |\xi| \lambda(t) \int_t^s \frac{\lambda(\tau)}{\lambda(s)} d\tau \leq |\xi| \frac{\lambda(t)}{\lambda(s)} \Lambda(t_\xi^{(1)}) \lesssim \frac{\lambda(t)}{\lambda(s)}.$$

Second column. For the second column we have $v(s, \xi) = 0$ and $w(s, \xi) = 1$. Plugging again the second integral equation into the first one implies

$$(2.33) \quad v(t, \xi) = -i|\xi|\lambda(t)(s-t) - |\xi|^2\lambda(t) \int_t^s \left(\int_t^\theta \lambda(\tau) d\tau \right) \omega^2(\theta) v(\theta, \xi) d\theta.$$

Therefore, we expect $v(t, \xi)/(|\xi|\lambda(t)(s-t)) \in L^\infty(Z'_{pd}(N))$. Rewriting the integral equation yields

$$(2.34) \quad \frac{iv(t, \xi)}{|\xi|\lambda(t)(s-t)} = 1 + \int_t^s k_2(t, \theta, \tau) \frac{iv(\theta)}{|\xi|\lambda(\theta)(s-\theta)} d\theta$$

with new kernel

$$(2.35) \quad k_2(t, \theta, \xi) = -|\xi|^2\lambda(\theta)\omega^2(\theta) \frac{s-\theta}{s-t} \int_t^\theta \lambda(\tau) d\tau.$$

Note that $|k_2(t, \theta, \xi)| \leq |k_1(t, \theta, \xi)|$ (from $t \leq \theta \leq s$) such that the kernel estimate

$$(2.36) \quad \sup_{(t, \xi) \in Z_{pd}} \int_0^s \sup_{0 \leq \tilde{t} \leq \theta} |k_2(\tilde{t}, \theta, \xi)| d\theta \lesssim 1$$

holds true, which in turn implies convergence of the corresponding Neumann series. Therefore, as claimed,

$$(2.37) \quad |v(t, \xi)| \lesssim |\xi|\lambda(t)(s-t)$$

and the second integral equation implies

$$(2.38) \quad |w(t, \xi)| \lesssim 1 + |\xi|^2\lambda(t) \int_t^s \lambda(\theta)(s-\theta) d\theta \leq 1 + |\xi|^2\lambda(t) \int_t^{t_\xi^{(1)}} (\Lambda(\tau) - 1) d\tau.$$

This is uniformly bounded due to assumption (A1). Indeed, the second term vanishes for $t = t_\xi^{(1)}$ and its derivative

$$\lambda'(t) \int_t^{t_\xi^{(1)}} (\Lambda(\tau) - 1) d\tau - \lambda(t)(\Lambda(t) - 1)$$

changes sign. At critical points we get the upper bound $|\xi|^2\lambda^2(t)\Lambda(t)/\lambda'(t) \lesssim |\xi|^2\Lambda^2(t) \lesssim 1$ due to the lower bound on $\lambda'(t)$ by (A1). \square

2.3. Consideration in the hyperbolic zone. We define implicitly $t_\xi^{(2)}$ by

$$(2.39) \quad \Theta(t_\xi^{(2)})|\xi| = N$$

and denote

$$(2.40) \quad Z_{hyp}(N) = \{(t, \xi) : t \geq t_\xi^{(2)}\}.$$

By (A3) we know that $t_\xi^{(2)} > t_\xi^{(1)}$ and $Z_{hyp}(N)$ lies on top of $Z_{pd}(N)$ with a gap in between. The consideration in the hyperbolic zone follows essentially [3] or [4]. Our aim is to obtain the statement of Lemma 2.1, but now for the true $\mathcal{E}(t, s, \xi)$ and in the smaller zone.

Lemma 2.4. *Assume (A1), (A2), (A4) and (A5). Then the fundamental solution satisfies*

$$(2.41) \quad \|\mathcal{E}(t, s, \xi)\| \approx \frac{\sqrt{\lambda(t)}}{\sqrt{\lambda(s)}}$$

uniformly in $Z_{hyp}(N)$.

Basically, we follow the proof of Lemma 2.1. The main difference is that the remainder terms satisfy worse estimates (due to the presence of $\omega(t)$ in the coefficient matrix), so we do not stop after the second step. We apply m steps instead. Before giving the proof we will give this diagonalisation procedure in detail.

We define the following symbol classes within $Z_{hyp}(N)$. We say that $a(t, \xi)$ belongs to $\mathcal{S}_N^\ell\{m_1, m_2, m_3\}$ if the symbol estimate

$$(2.42) \quad |D_t^k a(t, \xi)| \leq C_k |\xi|^{m_1} \lambda(t)^{m_2} \Xi(t)^{-m_3-k}$$

holds true for all $k = 0, 1, \dots, \ell$ and all $(t, \xi) \in Z_{hyp}(N)$. These symbol classes satisfy natural calculus rules. The most important ones for us are collected in the following proposition.

Proposition 2.5. (1) $\mathcal{S}_N^\ell\{m_1, m_2, m_3\}$ is a vector space;
 (2) $\mathcal{S}_N^\ell\{m_1, m_2, m_3\} \hookrightarrow \mathcal{S}_{N'}^{\ell'}\{m_1 + k, m_2 + k, m_3 - k\}$ if $N' \geq N$, $\ell' \leq \ell$ and $k \geq 0$;
 (3) $\mathcal{S}_N^\ell\{m_1, m_2, m_3\} \cdot \mathcal{S}_N^{\ell'}\{m'_1, m'_2, m'_3\} \hookrightarrow \mathcal{S}_N^\ell\{m_1 + m'_1, m_2 + m'_2, m_3 + m'_3\}$;
 (4) $D_t^k \mathcal{S}_N^\ell\{m_1, m_2, m_3\} \hookrightarrow \mathcal{S}_N^{\ell-k}\{m_1, m_2, m_3 + k\}$ for $k \leq \ell$;
 (5) $\mathcal{S}_N^0\{1 - m, 1 - m, m\} \hookrightarrow L_\xi^\infty L_t^1(Z_{hyp}(N))$ with m from assumption (A5).

Proofs are straightforward. The embedding relation (A2) follows essentially from our requirement $\lambda(t)\Xi(t) \gtrsim \Theta(t)$ in combination with the definition of the zone.

In order to solve (2.3) within $Z_{hyp}(N)$ we apply several transformations. In a first step we set $V^{(0)} = M^{-1}(t)V$ with

$$(2.43) \quad M(t) = \frac{1}{\omega(t)} \begin{pmatrix} 1 & -1 \\ \omega(t) & \omega(t) \end{pmatrix}, \quad M^{-1}(t) = \frac{1}{2} \begin{pmatrix} \omega(t) & 1 \\ -\omega(t) & 1 \end{pmatrix},$$

such that

$$(2.44) \quad D_t V^{(0)} = \left(\begin{pmatrix} \lambda(t)\omega(t)|\xi| & \\ & -\lambda(t)\omega(t)|\xi| \end{pmatrix} + \frac{D_t(\lambda(t)\omega(t))}{2\lambda(t)\omega(t)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right) V^{(0)}$$

holds true. Note that the coefficient function $a(t)$ appears in both expressions, such that the first (diagonal) matrix satisfies $\mathcal{D}_0(t, \xi) \in \mathcal{S}_N^m\{1, 1, 0\}$, while the second (remainder) term is of lower order in our symbol hierarchy $R_0(t, \xi) \in \mathcal{S}_N^{m-1}\{0, 0, 1\}$.

We set $\mathcal{D}_1(t, \xi) = \mathcal{D}_0(t, \xi) + \text{diag } R_0(t, \xi)$ and $R_1(t, \xi) = R_0(t, \xi) - \text{diag } R_0(t, \xi)$. Now we can improve the behaviour of this system within our symbol classes step by step.

Lemma 2.6. *There exists a zone constant N such that for all $k \leq m - 1$ we can find matrices*

- $N_k(t, \xi) \in \mathcal{S}_N^{m-k}\{0, 0, 0\}$, invertible and $N_k^{-1}(t, \xi) \in \mathcal{S}_N^{m-k}\{0, 0, 0\}$;

- $\mathcal{D}_k(t, \xi) \in \mathcal{S}_N^{m-k}\{1, 1, 0\}$ *diagonal and*
 $\mathcal{D}_k(t, \xi) = \text{diag}(\tau_k^+(t, \xi), \tau_k^-(t, \xi))$ *with* $|\tau_k^+(t, \xi) - \tau_k^-(t, \xi)| \gtrsim \lambda(t)|\xi|$;
- $R_k(t, \xi) \in \mathcal{S}_N^{m-k}\{1 - k, 1 - k, k\}$ *antidiagonal*

defined on $Z_{\text{hyp}}(N)$ such that the operator identity

$$(2.45) \quad (\mathcal{D}_t - \mathcal{D}_1(t, \xi) - R_1(t, \xi))N_k(t, \xi) = N_k(t, \xi)(\mathcal{D}_t - \mathcal{D}_{k+1}(t, \xi) - R_{k+1}(t, \xi))$$

holds true.

Proof. We construct the matrices $N_k(t, \xi)$ recursively as products

$$(2.46) \quad N_k(t, \xi) = \prod_{j=1}^k (\mathcal{I} + N^{(j)}(t, \xi))$$

of invertible matrices satisfying

$$(2.47) \quad (\mathcal{D}_t - \mathcal{D}_k(t, \xi) - R_k(t, \xi))(\mathcal{I} + N^{(k)}(t, \xi)) \\ = (\mathcal{I} + N^{(k)}(t, \xi))(\mathcal{D}_t - \mathcal{D}_{k+1}(t, \xi) - R_{k+1}(t, \xi)), \quad k + 1 \leq m - 1.$$

This is a straightforward generalisation of the second diagonalisation step in the proof of Lemma 2.1. Indeed, the matrices $\mathcal{D}_1(t, \xi)$ and $R_1(t, \xi)$ satisfy clearly the above statements. Assume now, the statements about $\mathcal{D}_k(t, \xi)$ and $R_k(t, \xi)$ are true. Then we can construct

$$(2.48) \quad N^{(k)}(t, \xi) = \begin{pmatrix} \frac{(R_k(t, \xi))_{12}}{\tau_k^+(t, \xi) - \tau_k^-(t, \xi)} \\ -\frac{(R_k(t, \xi))_{21}}{\tau_k^+(t, \xi) - \tau_k^-(t, \xi)} \end{pmatrix} \in \mathcal{S}_N^{m-k}\{-k, -k, k\},$$

such that $\mathcal{I} + N^{(k)}(t, \xi)$ is invertible for sufficiently large N (following directly from $\|N^{(k)}(t, \xi)\| \lesssim \frac{1}{|\xi|^k \lambda^k(t) \Xi^k(t)} \lesssim \frac{1}{|\xi|^k \Theta^k(t)} \leq \frac{1}{N^k} \rightarrow 0$ as $N \rightarrow \infty$). Furthermore, by construction

$$(2.49) \quad [\mathcal{D}_k(t, \xi), N_k(t, \xi)] + R_k(t, \xi) = 0,$$

such that

$$(2.50) \quad B^{(k)}(t, \xi) = (\mathcal{D}_t - \mathcal{D}_k(t, \xi) - R_k(t, \xi))(\mathcal{I} + N^{(k)}(t, \xi)) - (\mathcal{I} + N^{(k)}(t, \xi))(\mathcal{D}_t - \mathcal{D}_k(t, \xi)) \\ = \mathcal{D}_t N^{(k)}(t, \xi) - R_k(t, \xi) N^{(k)}(t, \xi) \in \mathcal{S}_N^{m-k-1}\{-k, -k, k + 1\}.$$

Setting

$$(2.51) \quad \mathcal{D}_{k+1}(t, \xi) = \mathcal{D}_k(t, \xi) - \text{diag}((\mathcal{I} + N^{(k)}(t, \xi))^{-1} B^{(k)}(t, \xi))$$

and

$$(2.52) \quad R_{k+1}(t, \xi) = -(\mathcal{I} + N^{(k)}(t, \xi))^{-1} B^{(k)}(t, \xi) + \text{diag}((\mathcal{I} + N^{(k)}(t, \xi))^{-1} B^{(k)}(t, \xi))$$

completes the construction and the symbol estimate of $B^{(k)}$ from (2.50) finally implies $|\tau_{k+1}^+(t, \xi) - \tau_{k+1}^-(t, \xi)| \leq |\tau_k^+(t, \xi) - \tau_k^-(t, \xi)| + \lambda(t)|\xi| \frac{C}{N}$. If we choose N large enough the statement is proven. \square

Lemma 2.7. *The diagonal entries satisfy*

$$(2.53) \quad \operatorname{Im} \tau_k^+(t, \xi) = \operatorname{Im} \tau_k^-(t, \xi) = -\frac{\lambda'(t)}{2\lambda(t)} - \frac{\omega'(t)}{2\omega(t)} - \sum_{j=1}^{k-1} \frac{\partial_t d_j(t, \xi)}{2(d_j(t, \xi) - 1)}$$

with $d_j(t, \xi) = -\det N^{(j)}(t, \xi)$ being real and $|d_j(t, \xi)| \leq c < 1$ uniform on $Z_{hyp}(N)$.

Proof. The proof goes by induction over k . We will show that the above statement and the following hypothesis

(H_k): $R_k(t, \xi)$ has the form $R_k = i(\frac{\bar{\beta}_k}{\beta_k})$ with complex-valued $\beta_k(t, \xi)$

are valid. For $k = 1$ the assertion (H₁) is clearly true with real-valued $\beta_1(t, \xi) = \frac{a'(t)}{2a(t)}$ and $\tau_1^\pm = \pm a(t)|\xi| - i\frac{a'(t)}{2a(t)}$ clearly satisfies the statement of Lemma 2.7.

We will show that (H_k) implies (H_{k+1}). The construction implies $N^{(k)} = \frac{i}{\delta_k}(\frac{\bar{\beta}_k}{\beta_k})$ with $\delta_k(t, \xi) = \tau_k^+(t, \xi) - \tau_k^-(t, \xi)$ being real and $|d_k(t, \xi)| = |\det N^{(k)}| = |\beta_k|^2/|\delta_k|^2 \leq c < 1$ (for our choice of the zone constant N). Following [3] we obtain

$$(2.54) \quad (I + N^{(k)})^{-1}(\mathcal{D}_k + R_k)(I + N^{(k)}) \\ = \frac{1}{1 - d_k} (\operatorname{diag}(\tau_k^+ - d_k \tau_k^+ - \delta_k d_k, \tau_k^- - d_k \tau_k^- + \delta_k d_k) + d_k R_k)$$

and

$$(2.55) \quad (I + N^{(k)})^{-1}(D_t N^{(k)}) = \frac{1}{1 - d_k} \left(\begin{pmatrix} i\frac{\bar{\beta}_k}{\delta_k} \partial_t \frac{\beta_k}{\delta_k} & \\ & i\frac{\beta_k}{\delta_k} \partial_t \frac{\bar{\beta}_k}{\delta_k} \end{pmatrix} + \begin{pmatrix} & -\partial_t \frac{\bar{\beta}_k}{\delta_k} \\ \partial_t \frac{\beta_k}{\delta_k} & \end{pmatrix} \right)$$

such that $\operatorname{Re} \frac{\beta_k}{\delta_k} \partial_t \frac{\bar{\beta}_k}{\delta_k} = \frac{\partial_t d_k}{2} = \operatorname{Re} \frac{\bar{\beta}_k}{\delta_k} \partial_t \frac{\beta_k}{\delta_k}$ implies

$$(2.56) \quad \tau_{k+1}^\pm = \tau_k^\pm \mp \frac{1}{1 - d_k} \left(d_k \delta_k + \operatorname{Im} \left(\frac{\beta_k}{\delta_k} \partial_t \frac{\bar{\beta}_k}{\delta_k} \right) \right) - i \frac{\partial_t d_k}{2(d_k - 1)}.$$

Hence δ_{k+1} is real again and R_{k+1} satisfies (H_{k+1}). Furthermore, the statement of Lemma 2.7 follows for $k + 1$. \square

Proof. (of Lemma 2.4) It is sufficient to solve the simpler system

$$(2.57) \quad D_t \mathcal{E}_m(t, s, \xi) = (\mathcal{D}_m(t, \xi) + R_m(t, \xi)) \mathcal{E}_m(t, s, \xi), \quad \mathcal{E}_m(s, s, \xi) = I.$$

Lemma 2.7 implies that the fundamental solution of the diagonal part,

$$(2.58) \quad \tilde{\mathcal{E}}_m(t, s, \xi) = \exp \left(i \int_s^t \mathcal{D}_m(\theta, \xi) d\theta \right) = \operatorname{diag} \left(e^{i \int_s^t \tau_m^+(\theta, \xi) d\theta}, e^{i \int_s^t \tau_m^-(\theta, \xi) d\theta} \right),$$

has condition number $\operatorname{cond} \tilde{\mathcal{E}}_m(t, s, \xi) = 1$. Therefore, we can make the *ansatz* $\mathcal{E}_m(t, s, \xi) = \tilde{\mathcal{E}}_m(t, s, \xi) \mathcal{Q}_m(t, s, \xi)$ and get for $\mathcal{Q}_m(t, s, \xi)$ the system

$$(2.59) \quad D_t \mathcal{Q}_m(t, s, \xi) = \mathcal{R}_m(t, s, \xi) \mathcal{Q}_m(t, s, \xi), \quad \mathcal{Q}_m(s, s, \xi) = I$$

with coefficient matrix $\mathcal{R}_m(t, s, \xi) = \tilde{\mathcal{E}}_m(s, t, \xi) R_m(t, \xi) \tilde{\mathcal{E}}_m(t, s, \xi)$ subject to the same bounds like $R_m(t, \xi)$,

$$(2.60) \quad \|\mathcal{R}_m(t, s, \xi)\| = \|R_m(t, \xi)\| \lesssim \frac{1}{|\xi|^{1-m} \lambda^{1-m}(t) \Xi^m(t)}.$$

Therefore, $\mathcal{Q}_m(t, s, \xi)$ can be represented as Peano-Baker series and satisfies the uniform estimate

$$(2.61) \quad \begin{aligned} \|\mathcal{Q}_m(t, s, \xi)\| &\leq \exp \left(\int_s^t \|\mathcal{R}_m(\theta, s, \xi)\| d\theta \right) \leq \exp \left(\int_{t_\xi^{(2)}}^\infty \frac{C}{|\xi|^{1-m} \lambda^{1-m}(\theta) \Xi^m(\theta)} d\theta \right) \\ &\leq \exp \left(\frac{C}{|\xi|^{1-m} \Theta^{1-m}(t_\xi^{(2)})} \right) \lesssim 1. \end{aligned}$$

Additionally, by Liouville theorem and the invariance of the trace under similarity transformations we get

$$(2.62) \quad \det \mathcal{Q}_m(t, s, \xi) = \exp \left(i \int_s^t \operatorname{tr} \mathcal{R}_m(\theta, s, \xi) d\theta \right) = \exp \left(i \int_s^t \operatorname{tr} R_m(\theta, \xi) d\theta \right) = 1$$

and $\|\mathcal{Q}_m^{-1}(t, s, \xi)\| \lesssim 1$. Thus, representing $\mathcal{E}(t, s, \xi)$ as

$$(2.63) \quad \mathcal{E}(t, s, \xi) = M(t) N_m(t, \xi) \tilde{\mathcal{E}}_m(t, s, \xi) \mathcal{Q}_m(t, s, \xi) N_m^{-1}(s, \xi) M^{-1}(s)$$

gives by the uniform bounds of (2.61) and Lemma 2.6

$$(2.64) \quad \|\mathcal{E}(t, s, \xi)\| \approx \|\tilde{\mathcal{E}}_m(t, s, \xi)\| = \exp \left(- \int_s^t \operatorname{Im} \tau_m^\pm(\theta, \xi) d\theta \right) \approx \frac{\sqrt{\lambda(t)}}{\sqrt{\lambda(s)}}$$

and the statement is proven. \square

Remark 2.2. Again, we have established much more than just the two-sided estimate of Lemma 2.4. We got a precise description of the structure of the fundamental solution $\mathcal{E}(t, s, \xi)$ for large time t . Indeed, like in Remark 2.1 about Lemma 2.1 we established that the transformation matrices $N_k(t, \xi) \rightarrow I$ and the amplitudes $\mathcal{Q}_m(t, s, \xi) \rightarrow \mathcal{Q}_m(\infty, s, \xi)$ as $t \rightarrow \infty$ locally uniform in $\xi \neq 0$. Therefore, solutions are determined for large time by $M(t) \tilde{\mathcal{E}}_m(t, s, \xi)$.

2.4. Consideration in the intermediate zone. This zone is defined as

$$(2.65) \quad Z_{int}(N) = \{(t, \xi) : t_\xi^{(1)} \leq t \leq t_\xi^{(2)}\}.$$

We want to relate $\mathcal{E}(t, s, \xi)$ to $\mathcal{E}_\lambda(t, s, \xi)$ within this zone. For this we employ the stabilisation condition *in combination* with Lemma 2.1.

Lemma 2.8. *Assume (A1) – (A3). Then the fundamental solution satisfies*

$$(2.66) \quad \|\mathcal{E}(t, s, \xi)\| \approx \frac{\sqrt{\lambda(t)}}{\sqrt{\lambda(s)}}$$

uniformly in $Z_{int}(N)$.

Proof. We make the *ansatz* $\mathcal{E}(t, s, \xi) = \mathcal{E}_\lambda(t, s, \xi) \mathcal{Q}_{int}(t, s, \xi)$. Then the matrix $\mathcal{Q}_{int}(t, s, \xi)$ satisfies the differential equation

$$(2.67) \quad D_t \mathcal{Q}_{int}(t, s, \xi) = \mathcal{E}_\lambda(s, t, \xi) (A(t, \xi) - A_\lambda(t, \xi)) \mathcal{E}_\lambda(t, s, \xi) \mathcal{Q}_{int}(t, s, \xi)$$

with initial condition $\mathcal{Q}_{int}(s, s, \xi) = I$. The stabilisation condition together with the uniform bound of the condition number $\text{cond } \mathcal{E}_\lambda(t, s, \xi) \lesssim 1$ from Lemma 2.1 implies that the coefficient matrix of this problem satisfies

$$(2.68) \quad \begin{aligned} & \int_{t_\xi^{(1)}}^{t_\xi^{(2)}} \|\mathcal{E}_\lambda(s, t, \xi) (A(t, \xi) - A_\lambda(t, \xi)) \mathcal{E}_\lambda(t, s, \xi)\| dt \\ & \lesssim \int_{t_\xi^{(1)}}^{t_\xi^{(2)}} \|A(t, \xi) - A_\lambda(t, \xi)\| dt \\ & \approx |\xi| \int_{t_\xi^{(1)}}^{t_\xi^{(2)}} \lambda(t) |\omega^2(t) - 1| dt \lesssim |\xi| \Theta(t_\xi^{(2)}) = N. \end{aligned}$$

Therefore, the representation of $\mathcal{Q}_{int}(t, s, \xi)$ as Peano-Baker series implies the uniform boundedness of $\mathcal{Q}_{int}(t, s, \xi)$ over the intermediate zone. Furthermore, we get $\det \mathcal{Q}_{int}(t, s, \xi) = 1$ from Liouville theorem and conclude that $\mathcal{Q}_{int}(t, s, \xi)$ is uniformly invertible. This transfers the two-sided estimate from $\mathcal{E}_\lambda(t, s, \xi)$ to $\mathcal{E}(t, s, \xi)$ and the statement is proven. \square

3. ENERGY INEQUALITIES

3.1. Estimates from above. The statements of Lemmata 2.4 and 2.8 imply that the energy $\mathbb{E}_\lambda(u; t)$ increases (for large t and ξ) like $\lambda(t)$. Our first aim is to combine this with the estimate from Lemma 2.2 / Corollary 2.3. For this we assume that

(A1₊): the coefficient function $\lambda(t)$ satisfies $t\sqrt{\lambda(t)} \lesssim \Lambda(t)$ in addition to (A1),

which is true in all example cases. This might even be a consequence of (A1), however we don't know that for certain.

Theorem 3.1. *Assume (A1₊) – (A5). Then all solutions $u(t, x)$ to the Cauchy problem (1.1) satisfy the a priori estimate*

$$(3.1) \quad \mathbb{E}_\lambda(u; t) \leq C \lambda(t) (\|u_1\|_{H^1}^2 + \|u_2\|_{L^2}^2)$$

with a constant C depending only on the coefficient function $a(t)$.

Proof. Corollary 2.3 implies the estimate

$$(3.2) \quad \|\mathcal{E}(t, 0, \xi) \text{diag}(|\xi|/\langle \xi \rangle, 1)\| \lesssim \max(t|\xi|\lambda(t), 1) \lesssim \sqrt{\lambda(t)}$$

in combination with (A1₊) and the definition of the pseudo-differential zone. By Lemmata 2.4 and 2.8

$$(3.3) \quad \begin{aligned} \|\mathcal{E}(t, 0, \xi) \text{diag}(|\xi|/\langle \xi \rangle, 1)\| & \lesssim \|\mathcal{E}(t, t_\xi^{(1)}, \xi)\| \|\mathcal{E}(t_\xi^{(1)}, 0, \xi) \text{diag}(|\xi|/\langle \xi \rangle, 1)\| \\ & \lesssim \sqrt{\lambda(t)}. \end{aligned}$$

follows for all $t \geq t_\xi^{(1)}$ and the proof is complete. \square

Remark 3.1. If $\lambda(t)$ is bounded we do not need to change the space for the data. The additional factor $|\xi|$ for small frequencies used in the previous argument to compensate the estimate of Corollary 2.3 is not necessary in this case and the statement

$$(3.4) \quad \mathbb{E}_\lambda(u; t) \lesssim \mathbb{E}_\lambda(u; 0), \quad \lambda(t) \leq c < \infty,$$

from [3] follows. However, if $\lambda(t)$ is unbounded, this estimate is in general false. This can be seen by constructing explicit representations in terms of special functions (like done for $a(t) = t^\ell$ in [6] or $a(t) = e^t$ in [7]) and evaluating them in the neighbourhood of $\xi = 0$. See also [8] for a similar argument in the dissipative case.

3.2. Bounds from below. Outside the pseudo-differential zone we already achieved lower bounds. Our strategy is to relate solutions to a quantity which can be controlled everywhere. This idea will be combined with an application of Banach-Steinhaus theorem on a dense subspace of $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *excluding* the exceptional frequency $\xi = 0$.

Theorem 3.2. *Assume (A1₊) – (A5). Then for all data $u_1 \in H^1(\mathbb{R}^n)$ and $u_2 \in L^2(\mathbb{R}^n)$ there exists a constant C such that*

$$(3.5) \quad \mathbb{E}_\lambda(u; t) \geq C\lambda(t)$$

holds true. (The constant C depends in a nontrivial way on the data.)

Proof. We proceed in two steps. In a first step we assume that the data u_1 and u_2 satisfy the condition $0 \notin \text{supp } \hat{u}_i$ (which directly implies $0 \notin \text{supp } \hat{u}(t, \cdot)$ for all $t \geq 0$). We want to compare $(|\xi|\hat{u}, D_t \hat{u})^T$ to $\hat{\mathcal{E}}(t, 0, \xi)(\tilde{w}_1, \tilde{w}_2)^T$, where

$$(3.6) \quad \hat{\mathcal{E}}(t, 0, \xi) = \begin{cases} \sqrt{\lambda(t_\xi^{(1)})} \mathcal{E}(t, t_\xi^{(1)}, \xi), & t \geq t_\xi^{(1)}, \\ \sqrt{\lambda(t)} \mathbf{I}, & 0 \leq t \leq t_\xi^{(1)}, \end{cases}$$

for suitably chosen $\tilde{w}_i \in L^2(\mathbb{R}^n)$. By definition and Lemmata 2.4 and 2.8 we have $\|\hat{\mathcal{E}}(t, 0, \xi)\| \approx \sqrt{\lambda(t)}$ and $\|\hat{\mathcal{E}}^{-1}(t, 0, \xi)\| \approx 1/\sqrt{\lambda(t)}$ such that the two-sided estimate

$$(3.7) \quad \|\hat{\mathcal{E}}(t, 0, \xi)(\tilde{w}_1, \tilde{w}_2)^T\|_2 \approx \sqrt{\lambda(t)}\|(\tilde{w}_1, \tilde{w}_2)\|_2$$

follows. Now we will construct \tilde{w}_i such that

$$(3.8) \quad \frac{1}{\sqrt{\lambda(t)}}\|(|\xi|\hat{u}, D_t \hat{u})^T - \hat{\mathcal{E}}(t, 0, \xi)(\tilde{w}_1, \tilde{w}_2)^T\|_2 \rightarrow 0, \quad t \rightarrow \infty.$$

Since $0 \notin \text{supp } \hat{u}(t, \cdot)$ the difference vanishes identically for sufficiently large t if we define

$$(3.9) \quad \begin{aligned} (\tilde{w}_1, \tilde{w}_2)^T &= \lim_{t \rightarrow \infty} \hat{\mathcal{E}}^{-1}(t, 0, \xi) \mathcal{E}(t, 0, \xi) \text{diag}(|\xi|/\langle \xi \rangle, 1)(\langle \xi \rangle \hat{u}_1, \hat{u}_2)^T \\ &= \frac{1}{\sqrt{\lambda(t_\xi^{(1)})}} \mathcal{E}(t_\xi^{(1)}, 0, \xi) \text{diag}(|\xi|/\langle \xi \rangle, 1)(\langle \xi \rangle \hat{u}_1, \hat{u}_2)^T \end{aligned}$$

and by the argument used in the previous proof the appearing multiplier is uniformly bounded in ξ . Thus for all data with $0 \notin \text{supp } \hat{u}_i$ we constructed $\tilde{w}_i \in L^2(\mathbb{R}^n)$.

In a second step we relax the condition on the data. This follows by Banach-Steinhaus theorem since we are already on a dense subset of $[L^2(\mathbb{R}^n)]^2$ and the left hand side of (3.8) is uniformly bounded by Theorem 3.1. Thus (3.8) holds for all solutions if we define \tilde{w}_i by (3.9) in terms of the data.

Finally, from (3.7) and (3.8) the desired statement follows. \square

4. EXAMPLES AND COUNTER-EXAMPLES

We will collect some examples for shape functions $\lambda(t)$ and perturbations $\omega(t)$ which are admissible in our context. At first we introduce several classes examples depending on the growth order of $\lambda(t)$ and give suitable $\Theta(t)$ and $\Xi(t)$ for assumptions (A1) to (A5). Later on we construct functions $\omega(t)$ subject to corresponding the bounds in all these cases.

Finally Section 4.3 is devoted to counter-examples, i.e. to show that the symbol-type assumption (A4'') for the coefficient is indeed sharp within certain classes of examples.

4.1. Classes of examples.

Example 4.1. (Polynomial growth) It is possible to choose all functions as polynomials. To be precise, we can set

$$(4.1a) \quad \lambda(t) = (1+t)^p,$$

$$(4.1b) \quad \Theta(t) = (1+t)^{1+q},$$

$$(4.1c) \quad \Xi(t) = (1+t)^r$$

for suitable choices of p, q and r . For any $p > 0$ assumption (A1) is fulfilled. Furthermore, we need $0 \leq q < p$ for (A3) and

$$(4.1d) \quad \begin{cases} 1 \geq r \geq r_m = 1 - p + q + \frac{p-q}{m}, & \text{for (A4')} \text{ and (A5')}, \\ 1 \geq r > r_\infty = 1 - p + q, & \text{for (A4'')} \text{ and (A5')}. \end{cases}$$

Increasing m makes r_m smaller and therefore the symbol condition (A4') becomes weaker for fixed derivatives (however, we need more derivatives). In this sense stabilisation allows to weaken symbol estimates.

Example 4.2. (Suprapolynomial growth) It is of interest to look at problems with faster increasing $\lambda(t)$. Therefore, we consider

$$(4.2a) \quad \lambda(t) = \exp(t^\alpha), \quad \alpha \in (0, 1)$$

$$(4.2b) \quad \Theta(t) = t^{-\beta} \exp(t^\alpha),$$

$$(4.2c) \quad \Xi(t) = t^\gamma.$$

Again we check all the requirements. Assumption (A1) is fulfilled. For (A3) we need $\beta > \alpha - 1$ and

$$(4.2d) \quad \begin{cases} 1 - \alpha \geq \gamma \geq \gamma_m = -\beta + \frac{\beta - \alpha + 1}{m}, & \text{for (A4')} \text{ and (A5')}, \\ 1 - \alpha \geq \gamma > \gamma_\infty = -\beta, & \text{for (A4'')} \text{ and (A5')}. \end{cases}$$

Again increasing m decreases γ_m and the interesting values for γ are negative.

Example 4.3. (Exponential growth) It is not essential that $\Xi(t)$ is polynomial. We can also consider

$$(4.3a) \quad \lambda(t) = e^t$$

$$(4.3b) \quad \Theta(t) = e^{at},$$

$$(4.3c) \quad \Xi(t) = e^{bt}$$

under suitable conditions on a and b . Assumption (A1) is fulfilled. For (A3) we need $a < 1$ and

$$(4.3d) \quad \begin{cases} 0 \geq b \geq b_m = a - 1 + \frac{1-a}{m}, & \text{for (A4')} \text{ and (A5')}, \\ 0 \geq b > b_\infty = a - 1, & \text{for (A4'')} \text{ and (A5')}. \end{cases}$$

4.2. Construction of admissible $\omega(t)$. Nontrivial examples for perturbations $\omega(t)$ of the ‘nice’ coefficient $\lambda(t)$ can be constructed in all cases. Our method depends on the choice of three positive sequences,

$$(4.4) \quad t_j \rightarrow \infty, \quad \delta_j \leq \Delta t_j = t_{j+1} - t_j \quad \text{and} \quad \eta_j \leq 1$$

and a function $\psi \in C_0^m(\mathbb{R})$ with

$$(4.5) \quad \text{supp } \psi \subseteq [0, 1], \quad -1 < \psi(t) < 1 \quad \text{and} \quad \int_0^1 |\psi(t)| dt = \frac{1}{2}.$$

Using these ingredients we define

$$(4.6) \quad \omega(t) = 1 + \sum_{j=1}^{\infty} \eta_j \psi\left(\frac{t - t_j}{\delta_j}\right),$$

the sum is converging trivially, since for each t at most one term is present. Furthermore, if $c_1 = \min \psi(t)$ and $c_2 = \max \psi(t)$ then we get the bound $0 < 1 + c_1 \leq \omega(t) \leq 1 + c_2$. It remains to look at the stabilisation properties and the symbol estimates. For the first one note that

$$(4.7) \quad \int_0^t \lambda(s) |\omega(s) - 1| ds = \sum_{j=1}^k \eta_j \int_{t_j}^{t_{j+1}} \lambda(s) \left| \psi\left(\frac{s - t_j}{\delta_j}\right) \right| ds \leq \sum_{j=1}^k \eta_j \delta_j \lambda(t_{j+1})$$

for $t \in [t_k, t_{k+1}]$. Similarly, we get the lower bound $\sum_{j=1}^k \eta_j \delta_j \lambda(t_j)$. Stabilisation property (A3) is ensured, if $\eta_j \delta_j$ are small enough to guarantee

$$(4.8) \quad \Theta(t_{k+1}) \approx \sum_{j=1}^k \eta_j \delta_j \lambda(t_{j+1}) \ll \sum_{j=1}^k \lambda(t_j) \Delta t_j \leq \Lambda(t_{k+1}).$$

Derivatives of $\omega(t)$ can be estimated by a multiplication with δ_j^{-1} on $[t_j, t_{j+1}]$, such that $\Xi(t)$ should satisfy $\Xi(t_j) \lesssim \delta_j$.

Example 4.4. (Polynomial case) We consider $\lambda(t) = (1 + t)^p$ from Example 4.1 and give a suitable choice of sequences. We choose $t_j = 2^j$, such that $\Delta t_j = 2^{j-1}$ and parameters

p , q and r from Example 4.1. Then δ_j is determined by $\delta_j \approx \Xi(t_j)$ as $\delta_j = 2^{jr-1}$ and (4.8) implies our choice for η_j ,

$$(4.9) \quad \eta_j = 2^{j(1+q-p-r)}.$$

Due to $r \geq r_m = 1 + p - q + (p - q)/m$ this choice implies $0 < \eta_j \leq 1$.

Example 4.5. (Suprapolynomial case) We consider $\lambda(t) = \exp(t^\alpha)$ from Example 4.2. To simplify the summation in (4.8) we adjust t_j such that $\lambda(t_j) \approx e^j$. This gives $t_j = j^{1/\alpha}$, $\Delta t_j \geq \frac{1}{\alpha} j^{1/\alpha-1}$. We choose $\delta_j = j^{\gamma/\alpha}$ (which is smaller than Δt_j due to $\gamma < 1 - \alpha$) and $\eta_j = j^{-(\beta+\gamma)/\alpha}$, such that the left part of (4.8) is satisfied.

Example 4.6. (Exponential case) We consider $\lambda(t) = e^t$ from Example 4.3. In this situation we choose $t_j = j$ and determine the sequences in dependence of the given parameters a and b from Example 4.3. This implies $\delta_j = e^{bj}$ and $\eta_j = e^{j(a-b-1)}$. By assumption $b < 0$ and $a - b - 1 \leq 0$ and therefore $\delta_j < 1$ and $\eta_j \leq 1$.

4.3. Counter-examples. Finally we want to apply a modified Floquet approach to show that our considerations are optimal in the sense that for given $\lambda(t)$ from our example classes there exists a coefficient $\omega(t)$ which violates one of the assumptions nearly and in turn leads to the non-existence of uniform bounds. The approach is a generalisation of considerations from [9], [10] and implicitly also used in [3].

The construction of the coefficient function follows that from the previous section with one alteration, we do not just add one bump $\psi(t)$ in the intervals $[t_j, t_{j+1}]$ but ν_j many of them. Thus we are given sequences t_j , δ_j subject to (4.4) and $\nu_j \in \mathbb{N}$ together with a real-valued function $\psi \in C_0^\infty[0, 1]$ subject to (4.5) and 1-periodised as $b(t) = \psi(t \bmod 1)$. Then $\omega(t)$ is given by

$$(4.10) \quad \omega(t) = \begin{cases} 1, & t \notin \bigcup_{j=1}^\infty [t_j, t_j + \delta_j], \\ 1 + b\left(\frac{\nu_j}{\delta_j}(t - t_j)\right), & t \in [t_j, t_j + \delta_j]. \end{cases}$$

All parameters are adjusted in a suitable way in dependence of the given $\lambda(t)$. Stabilisation is guaranteed if $\Theta(t_{k+1}) \approx \sum_{j=1}^k \delta_j \lambda(t_{j+1})$ is small compared to $\Lambda(t_{k+1})$ and derivatives behave like multiplication with ν_k/δ_k on $[t_k, t_k + \delta_k]$, i.e. we have to impose $\Xi(t) \lesssim \delta_k/\nu_k$ for $t \in [t_k, t_k + \delta_k]$. By adjusting the sequence δ_j we can influence the stabilisation rate, while adjusting ν_j allows to change the symbolic estimates.

4.3.1. A lower estimate for the fundamental solution on $[t_j, t_j + \delta_j]$. We introduce a new local time-variable s such that $t(s) = t_j + s\delta_j/\nu_j$, $s \geq 0$, and look for the fundamental solution $\mathcal{Y}_j(s, s_0, \xi) := \mathcal{E}(t(s), t(s_0), \xi)$. This matrix-valued function satisfies

$$(4.11) \quad D_s \mathcal{Y}_j(s, s_0, \xi) = A_j(s, \xi) \mathcal{Y}_j(s, s_0, \xi), \quad \mathcal{Y}_j(s_0, s_0, \xi) = I$$

with coefficient matrix

$$(4.12) \quad A_j(s, \xi) = \frac{\delta_j}{\nu_j} A(t(s), \xi) = \frac{\delta_j}{\nu_j} \begin{pmatrix} -i \frac{\lambda'(t(s))}{\lambda(t(s))} & \lambda(t(s))|\xi| \\ \lambda(t(s))(1 + b(s))^2|\xi| & \end{pmatrix}, \quad s \in [0, \nu_j].$$

Our strategy is to relate this to the j -independent periodic problem with coefficient matrix

$$(4.13) \quad B(s, \tilde{\lambda}) = \begin{pmatrix} & \tilde{\lambda} \\ \tilde{\lambda}(1+b(s))^2 & \end{pmatrix}, \quad s \in \mathbb{R},$$

and parameter $\tilde{\lambda} = \delta_j \lambda(t_j) |\xi| / \nu_j$, i.e. to consider

$$(4.14) \quad D_s \mathcal{X}(s, \tilde{\lambda}) = B(s, \tilde{\lambda}) \mathcal{X}(s, \tilde{\lambda}), \quad \mathcal{X}(0, \tilde{\lambda}) = I.$$

Periodicity of the problem allows to restrict most considerations to the monodromy matrix $\mathcal{X}(\tilde{\lambda}) = \mathcal{X}(1, \tilde{\lambda})$. An elementary application of Floquet theory (based on $1+b(s)$ strictly positive, $b \in C^2(\mathbb{R})$, 1-periodic and real-valued) implies

Lemma 4.1 (Floquet theorem, cf. [11]). *There exists a bounded open subinterval \mathcal{I} of $(0, \infty)$ such that the monodromy matrix $\mathcal{X}(\tilde{\lambda})$ of (4.14) has for all parameters $\tilde{\lambda} \in \mathcal{I}$ a purely imaginary eigenvalue of magnitude larger than 1.*

Thus in order to get the worst possible behaviour of solutions we restrict our considerations to

$$(4.15) \quad \xi \in \Omega_j := \{\xi \in \mathbb{R}^n : \tilde{\lambda} = \frac{\delta_j \lambda(t_j)}{\nu_j} |\xi| \in \mathcal{I}\}.$$

It is evident that Ω_j is of positive measure, even if we shrink \mathcal{I} in such a way that we have a uniform lower bound for the magnitude of the eigenvalue. We use Lemma 4.1 to show that the following statement holds true for $\mathcal{Y}_j(\nu_j, 0, \xi)$ uniform in j and $\xi \in \Omega_j$.

Lemma 4.2. *Assume $\delta_j \frac{\lambda(t_j)}{\Lambda(t_j)} \rightarrow 0$, $\lambda(t_j + \delta_j) \approx \lambda(t_j)$ and $\Lambda(t_j + \delta_j) \approx \Lambda(t_j)$ uniform in j . Then there exists $\mu > 1$ depending on $b(s)$ and the choice of \mathcal{I} , such the matrix $\mathcal{Y}_j(\nu_j, 0, \xi)$ has for all $\xi \in \Omega_j$ and sufficiently large j an eigenvalue of modulus greater than $\frac{\mu^{\nu_j}}{2}$.*

Proof. Step 1. We write $\mathcal{Y}_j(\nu_j, 0, \xi) = \mathcal{Y}_j(\nu_j, \nu_j - 1, \xi) \cdots \mathcal{Y}_j(2, 1, \xi) \mathcal{Y}_j(1, 0, \xi)$ and prove the estimates

$$(4.16) \quad \|\mathcal{Y}_j(k+1, k, \xi) - \mathcal{Y}_j(k, k-1, \xi)\| \lesssim \frac{\delta_j}{\nu_j} \frac{\lambda(t_j)}{\Lambda(t_j)},$$

$$(4.17) \quad \|\mathcal{Y}_j(k+1, k, \xi) - \mathcal{X}(\tilde{\lambda})\| \lesssim \delta_j \frac{\lambda(t_j)}{\Lambda(t_j)}, \quad \tilde{\lambda} = \delta_j \lambda(t_j) |\xi| / \nu_j,$$

for $k = (0, 1, \dots, \nu_j - 1)$ uniform in $\xi \in \Omega_j$ and j . Note for this, that uniform in j , $\tau \in [0, 1]$ and k in the above stated ranges

$$\begin{aligned}
 & \|A_j(k + \tau, \xi) - A_j(k + \tau - 1, \xi)\| \approx \frac{\delta_j}{\nu_j} |\lambda(t(k + \tau)) - \lambda(t(k - 1 + \tau))| |\xi| \\
 & + \frac{\delta_j}{\nu_j} \left| \frac{\lambda'(t(k + \tau))}{\lambda(t(k + \tau))} - \frac{\lambda'(t(k - 1 + \tau))}{\lambda(t(k - 1 + \tau))} \right| \lesssim \frac{\delta_j^2}{\nu_j^2} |\xi| \lambda'(t(\zeta)) + \frac{\delta_j^2}{\nu_j^2} \left(\frac{\lambda(t(\zeta))}{\Lambda(t(\zeta))} \right)^2 \\
 (4.18) \quad & \lesssim \frac{\delta_j^2}{\nu_j^2} |\xi| \frac{\lambda^2(t_j)}{\Lambda(t_j)} \approx \frac{\delta_j}{\nu_j} \frac{\lambda(t_j)}{\Lambda(t_j)}
 \end{aligned}$$

$$\begin{aligned}
 & \|A_j(k + \tau, \xi) - B(\tau, \tilde{\lambda})\| \approx \frac{\delta_j}{\nu_j} |\lambda(t(k + \tau)) - \lambda(t(0))| |\xi| + \frac{\delta_j}{\nu_j} \frac{\lambda'(t(k + \tau))}{\lambda(t(k + \tau))} \\
 (4.19) \quad & \lesssim \frac{\delta_j^2}{\nu_j^2} |\xi| (k \lambda'(t(\zeta)) + \lambda'(t(k + \tau))) \lesssim \delta_j \frac{\lambda(t_j)}{\Lambda(t_j)},
 \end{aligned}$$

hold true (with intermediate values $\zeta \in [k - 1 + \tau, k + \tau]$ or $\zeta \in [0, k + \tau]$, respectively). By relative compactness of \mathcal{I} we know that $\|\mathcal{X}(s, \tilde{\lambda})\| \lesssim 1$ uniformly in s and $\tilde{\lambda} \in \mathcal{I}$. Thus, integration over τ gives the desired bounds (4.17),

$$(4.20) \quad \|\mathcal{Y}_j(k + 1, k, \xi) - \mathcal{X}(\tilde{\lambda})\| \leq \int_0^1 \|\mathcal{X}(\tau, \tilde{\lambda})\| \|A_j(k + \tau, \xi) - B(\tau, \tilde{\lambda})\| d\tau \lesssim \delta_j \frac{\lambda(t_j)}{\Lambda(t_j)}$$

uniform in k, j and ξ and using $\|\mathcal{Y}_j(k + \tau, k, \xi)\| \lesssim 1$, $\tau \in [0, 1]$, as consequence of $\nu_j \frac{\lambda(t_j)}{\Lambda(t_j)} \lesssim 1$ also (4.16),

$$\begin{aligned}
 & \|\mathcal{Y}_j(k + 1, k, \xi) - \mathcal{Y}_j(k, k - 1, \xi)\| \\
 (4.21) \quad & \leq \int_0^1 \|\mathcal{Y}_j(k - 1 + \tau, k - 1, \xi)\| \|A_j(k + \tau, \xi) - A_j(k - 1 + \tau, \xi)\| d\tau \lesssim \frac{\delta_j}{\nu_j} \frac{\lambda(t_j)}{\Lambda(t_j)}
 \end{aligned}$$

uniform in k, j and ξ .

Step 2. In a second step we want to compare $\mathcal{Y}_j(\nu_j, 0, \xi)$ with $\mathcal{X}^{\nu_j}(\tilde{\lambda})$. For this we denote by $M_{j,k}(\xi)$ diagonaliser of $\mathcal{Y}_j(k, k - 1, \xi)$ and $M(\tilde{\lambda})$ of $\mathcal{X}(\tilde{\lambda})$ which are of bounded condition uniform in j and close to each other. Furthermore, we denote by $D_{j,k}(\xi)$ and $D(\tilde{\lambda})$ the corresponding diagonal matrices (having the big eigenvalue as upper left corner entry). Then

$$\begin{aligned}
 & M^{-1}(\tilde{\lambda}) \mathcal{Y}_j(\nu_j, 0, \xi) M(\tilde{\lambda}) \\
 & = M^{-1}(\tilde{\lambda}) \mathcal{Y}_j(\nu_j, \nu_j - 1, \xi) \cdots \mathcal{Y}_j(2, 1, \xi) \mathcal{Y}_j(1, 0, \xi) M(\tilde{\lambda}) \\
 & = M^{-1}(\tilde{\lambda}) M_{j,\nu_j}(\xi) D_{j,\nu_j}(\xi) M_{j,\nu_j}^{-1}(\xi) M_{j,\nu_j-1}(\xi) \cdots D_{j,1}(\xi) M_{j,1}^{-1}(\xi) M(\tilde{\lambda}) \\
 (4.22) \quad & = (I + G_{j,\nu_j+1}(\xi)) D(\tilde{\lambda}) (I + G_{j,\nu_j}(\xi)) \cdots D(\tilde{\lambda}) (I + G_{j,2}(\xi)) D(\tilde{\lambda}) (I + G_{j,1}(\xi)),
 \end{aligned}$$

where $G_{j,k}(\xi) = D^{-1}(\tilde{\lambda}) D_{j,k}(\xi) M_{j,k}^{-1}(\xi) M_{j,k-1}(\xi) - I$ and for convenience $M_{j,0}(\xi) = M(\tilde{\lambda}) = M_{j,\nu_j+1}(\xi)$, $D_{j,\nu_j+1}(\xi) = D(\tilde{\lambda})$.

We need to look at the diagonaliser in more detail. Due to Liouville theorem we know that $\det \mathcal{Y}_j(k, k - 1, \xi) = \det \mathcal{X}(\tilde{\lambda}) = 1$ and the matrices $M_{j,k}(\xi)$ may be expressed in

terms of the entries of $\mathcal{Y}_j(k, k-1, \xi)$ and their eigenvalues. If we denote them as $y_{mn}^{(j,k)}(\xi)$ and the eigenvalues as $\mu_{j,k}^{\pm 1}(\xi)$ and assume for simplicity that $|y_{11}^{(j,k)}| \leq |y_{22}^{(j,k)}|$ a suitable diagonaliser is

$$(4.23) \quad M_{j,k}(\xi) = \begin{pmatrix} \frac{y_{21}^{(j,k)}}{\mu_{j,k}^{-1} - y_{22}^{(j,k)}} & 1 \\ 1 & \frac{y_{12}^{(j,k)}}{\mu_{j,k} - y_{11}^{(j,k)}} \end{pmatrix}.$$

A similar formula holds for $M(\tilde{\lambda})$. Due to the estimates of Step 1 a short calculation implies that $M_{j,k}^{-1}(\xi)M_{j,k-1}(\xi)$ approximates the identity,

$$(4.24) \quad \|M_{j,k}^{-1}(\xi)M_{j,k-1}(\xi) - \mathbf{I}\| \lesssim \frac{\delta_j \lambda(t_j)}{\nu_j \Lambda(t_j)}$$

uniform in j , and therefore

$$(4.25) \quad \|G_{j,k}(\xi)\| \lesssim \frac{\delta_j \lambda(t_j)}{\nu_j \Lambda(t_j)}, \quad k = 2, \dots, \nu_j,$$

and similarly

$$(4.26) \quad \|G_{j,1}(\xi)\|, \|G_{j,\nu_j+1}(\xi)\| \lesssim \delta_j \frac{\lambda(t_j)}{\Lambda(t_j)} \lesssim 1.$$

Therefore, the right hand side of (4.22) can be written as $D^{\nu_j}(\tilde{\lambda})$ plus a remainder of size

$$(4.27) \quad \lesssim \sum_{k=1}^{\nu_j-1} \binom{\nu_j-1}{k} \mu^{\nu_j-k} \left(\frac{\delta_j \lambda(t_j)}{\nu_j \Lambda(t_j)} \right)^k = \mu^{\nu_j} \left(\left(1 + \frac{1}{\mu} \frac{\delta_j \lambda(t_j)}{\nu_j \Lambda(t_j)} \right)^{\nu_j-1} - 1 \right)$$

uniformly in j and with $\|D(\tilde{\lambda})\| \sim \mu$. Due to our assumption $\delta_j \lambda(t_j)/\Lambda(t_j) \rightarrow 0$ and therefore the expression in brackets behaves like $\exp(\mu^{-1} \frac{\nu_j-1}{\nu_j} \delta_j \lambda(t_j)/\Lambda(t_j)) - 1$, which tends to zero as j approaches infinity. Choosing j large enough to bound this expression by $1/2$ and application of Bauer-Fike theorem proves the desired statement. \square

4.3.2. Choice of sequences. We assume j is large enough. Then the eigenvalues of $\mathcal{Y}_j(\nu_j, 0, \xi)$ are distinct and we are allowed to choose $V_j(\xi) \in C_0^\infty(\Omega_j)$ in such a way that it is normalised in L^2 -sense and coincides for each fixed ξ on its support with an eigenvector of $\mathcal{Y}_j(\nu_j, 0, \xi)$ corresponding to the large eigenvalue. Then we solve $D_t V = A(t, \xi)V$ with $V(t_j, \xi) = V_j(\xi)$ and denote by u_j the corresponding solution of the original problem. This yields a sequence of solutions with a remarkable property. As consequence of Lemma 4.2 we obtain uniformly in j , j large,

$$(4.28) \quad \mathbb{E}_\lambda(u_j; t_j + \delta_j) \gtrsim \mu^{2\nu_j} \mathbb{E}_\lambda(u_j; t_j) = \mu^{2\nu_j}.$$

This estimate contradicts with the estimate of Lemma 2.4, which implies uniform in j , j large,

$$(4.29) \quad \mathbb{E}_\lambda(u_j; t_j + \delta_j) \lesssim \frac{\lambda(t_j)}{\lambda(t_j + \delta_j)} \mathbb{E}_\lambda(u_j; t_j) \approx 1,$$

provided that $\Theta(t_j)|\xi| \approx \frac{\nu_j \Theta(t_j)}{\delta_j \lambda(t_j)} \rightarrow \infty$, i.e., $[t_j, t_j + \delta_j] \times \Omega_j$ belongs to the hyperbolic zone for large j . The estimates (4.28) and (4.29) contradict each other.

Thus, if we manage to construct sequences t_j , δ_j and ν_j such that all requirements are satisfied, a counter-example is found. We will do this for all our example classes.

Example 4.7. (Counter-example, polynomial case) Let $\lambda(t) = (1+t)^p$ for some $p \geq 0$ and $\Theta(t) = (1+t)^q$, $-1 \leq q < p$. We construct admissible sequences such that (A1)–(A3) hold, but (A4'') is violated in the sense that such an estimate holds only for a given arbitrarily small *negative* exponent.

We choose $t_j = 2^j$, $\delta_j = 2^{j(q-p+1)-1}$ and $\nu_j = \lceil 2^{j\epsilon(p-q)} \rceil$. Stabilisation is ensured and (A1) – (A3) are valid. By construction $\lambda(t_j + \delta_j) \approx \lambda(t_j)$ and $\Lambda(t_j + \delta_j) \approx \Lambda(t_j)$ holds uniformly in j and $\delta_j \lambda(t_j) / \Lambda(t_j) \approx 2^{j(q-p)} \rightarrow 0$. Thus, Lemma 4.2 can be applied. It remains to check the geometry restriction arising from the zone. It follows on Ω_j that $\Theta(t_j)|\xi| \approx \frac{\nu_j \Theta(t_j)}{\delta_j \lambda(t_j)} \approx 2^{j\epsilon(p-q)} \rightarrow \infty$.

We check how closely (A4'') is violated. Since derivatives behave like multiplications with ν_j/δ_j on the interval $[t_j, t_j + \delta_j]$, the best possible choice of $\Xi(t)$ would be

$$(4.30) \quad \Xi(t) = (1+t)^{q-p+1-\epsilon(p-q)} = \left(\frac{\lambda(t)}{\Theta(t)} \left(\frac{\Theta(t)}{\Lambda(t)} \right)^{-\epsilon} \right)^{-1}$$

in contrast to (A4'').

Example 4.8. (Counter-example, supra-polynomial case) Let $\lambda(t) = \exp(t^\alpha)$ with $\alpha \in (0, 1)$ and $\Theta(t) = t^{-\beta} \exp(t^\alpha)$, $\beta \geq \alpha - 1$. We choose $t_j = j^{1/\alpha}$ and $\delta_j = j^{-\beta/\alpha}$, such (A1)–(A3) are valid. Furthermore, we choose ν_j as $\nu_j = \lceil j^{\epsilon(\beta-\alpha+1)/\alpha} \rceil$, $\epsilon > 0$. It is evident that $\lambda(t_j + \delta_j) \approx \lambda(t_j)$ holds and similarly for the primitive. Again by construction $\delta_j \lambda(t_j) / \Lambda(t_j) \approx j^{-(\beta-\alpha+1)/\alpha}$ tends to zero if $\beta > \alpha - 1$ such that Lemma 4.2 applies. Furthermore, $\frac{\nu_j \Theta(t_j)}{\delta_j \lambda(t_j)} \approx j^{-\epsilon(\beta-\alpha+1)/\alpha} \rightarrow \infty$ and the counter-example is constructed. Derivatives behave like multiplication with $(1+t)^{\beta+\epsilon(\beta-\alpha+1)}$, i.e. (A4'') with exponent $-\epsilon$ (cf. equation (4.30)).

Example 4.9. (Counter-example, exponential case) Let $\lambda(t) = e^t$ and $\Theta(t) = e^{at}$, $a < 1$. We choosing $t_j = j$, $\delta_j = e^{j(a-1)}$ and $\nu_j = \lceil e^{j\epsilon(1-a)} \rceil$, $\epsilon > 0$. Then (A1)–(A3) hold. From $\delta_j \rightarrow 0$ we conclude $\lambda(t_j + \delta_j) \approx \lambda(t_j)$, the primitive is the same function. Furthermore, $\delta_j \lambda(t_j) / \Lambda(t_j) \approx \delta_j \rightarrow 0$ and Lemma 4.2 applies and the geometry restriction $\frac{\nu_j \Theta(t_j)}{\delta_j \lambda(t_j)} \approx e^{j\epsilon(1-a)} \rightarrow \infty$ is valid. The behaviour of derivatives is described by $\Xi(t) = e^{(a-1-\epsilon(1-a))t}$, thus again (A4'') holds only with exponent $-\epsilon$ (cf. equation (4.30)).

Hence, in all cases there exists a coefficient function $a(t)$ satisfying (A1) – (A3) and violating (A4'') to arbitrary small order for which the statement of Lemma 2.4 is false.

We finally want to discuss how to conclude a counter-example for the estimate of Theorem 3.1. For this we use the same idea as above, but estimate the corresponding Cauchy data on the level $t = 0$. Let for this V_j and u_j be constructed as above, $\mathbb{E}(u_j; t_j) = 1$ and $\mathbb{E}(u_j; t_j + \delta_j) \gtrsim \mu^{2\nu_j}$ uniform in the sequence u_j and assume that (A1)–(A3) hold

true. We are going to estimate

$$(4.31) \quad \mathcal{E}(0, t_j, \xi) V_j(\xi) = \mathcal{E}(0, t_\xi^{(1)}, \xi) \mathcal{E}(t_\xi^{(1)}, t_\xi^{(2)}, \xi) \mathcal{E}(t_\xi^{(2)}, t_j, \xi) V_j(\xi).$$

The first two factors satisfy Lemma 2.2 and 2.8, respectively. Both Lemmata are true as consequence of the above assumptions. For the third one we use

$$(4.32) \quad \|\mathcal{E}(t_\ell + \delta_\ell, t_{\ell+1}, \xi)\| \approx \frac{\sqrt{\lambda(t_{\ell+1})}}{\sqrt{\lambda(t_\ell + \delta_\ell)}}$$

uniform in ℓ with $\Lambda(t_\ell)|\xi| \geq N$ as consequence of Lemma 2.1 in combination with

$$(4.33) \quad \|\mathcal{E}(t_\ell, t_\ell + \delta_\ell, \xi)\| \lesssim e^{c\nu_\ell}, \quad c = \sup_\tau \frac{|b'(\tau)|}{1 + b(\tau)},$$

following from Gronwall inequality. Combining all these estimates we get for the Cauchy data $u_{j,1}$ and $u_{j,2}$ corresponding to the solution u_j

$$(4.34) \quad \|u_{j,1}\|_{H^1} + \|u_{j,2}\|_{L^2} \lesssim S^{j/2} \frac{t_\xi^{(1)} \sqrt{\lambda(t_\xi^{(1)})}}{\sqrt{\lambda(t_j)}} \exp\left(c \sum_{\ell=\ell_0}^{j-1} \nu_\ell\right),$$

where $S = \sup_j \lambda(t_j + \delta_j)/\lambda(t_j)$. Using (A1₊) and the definition of \mathcal{I}_j it follows that $t_\xi^{(1)} \sqrt{\lambda(t_\xi^{(1)})} \lesssim \Lambda(t_\xi^{(1)}) \approx |\xi|^{-1} \approx \delta_j \lambda(t_j)/\nu_j$. If Theorem 3.1 would be true, it would imply

$$(4.35) \quad \mu^{2\nu_j} \lesssim \mathbb{E}_\lambda(t_j + \delta_j; u_j) \lesssim \frac{\delta_j^2}{\nu_j^2} \lambda^2(t_j) S^j \exp\left(2c \sum_{\ell=\ell_0}^{j-1} \nu_\ell\right).$$

This gives a contradiction if

$$(4.36) \quad \frac{\delta_j^2}{\nu_j^2} \lambda^2(t_j) S^j \exp\left(2c \sum_{\ell=\ell_0}^{j-1} \nu_\ell - 2\nu_j \log \mu\right) \rightarrow 0, \quad j \rightarrow \infty.$$

We are going to check this for the previously constructed counter-example in the polynomial case.

Example 4.10. (Counter-example, polynomial case) We follow Example 4.7 for $\lambda(t) = (1+t)^p$, $\Theta(t) = (1+t)^q$, however with a minor change. We choose sequences $t_j = \sigma^j$, $\delta_j = \sigma^{j(q-p+1)-1}$ and $\nu_j = \lceil \sigma^{j\epsilon(p-q)} \rceil$, $\epsilon > 0$. All the previous considerations and conditions transfer, thus choosing ϵ small enough will closely violate (A4''). If we now consider the condition (4.36), the first factors increase exponentially like $S^j \sigma^{j(q+1-\epsilon(p-q))}$, while the second exponential can be estimated by

$$(4.37) \quad \exp\left(2c \frac{\sigma^{j\epsilon(p-q)} - 1}{\sigma^{\epsilon(p-q)} - 1} - 2\sigma^{j\epsilon(p-q)} \log \mu\right) \lesssim \exp(-c' \sigma^{j\epsilon(p-q)}), \quad c' > 0,$$

provided σ is chosen large, $c/(\sigma^{\epsilon(p-q)} - 1) < \log \mu$. Thus we obtain a counter-example to Theorem 3.1.

Example 4.11. (Counter-example, exponential case) We follow Example 4.9 with $\lambda(t) = e^t$, $\Theta(t) = e^{at}$ and choose the sequences $t_j = \sigma j$, $\delta_j = e^{\sigma j(a-1)}$ and $\nu_j = \lceil e^{\sigma j\epsilon(1-a)} \rceil$ with a new additional parameter σ . For any choice of $\sigma > 0$ the reasoning of Example 4.9 remains true. Furthermore, (4.36) follows provided that σ is chosen big enough, i.e. if $c/(e^{\sigma\epsilon(1-a)} - 1) < \log \mu$ holds.

REFERENCES

- [1] M. Reissig, K. Yagdjian, About the influence of oscillations on Strichartz-type decay estimates, *Rend. Semin. Mat., Torino* 58 (3) (2000) 375–388.
- [2] M. Reissig, L_p - L_q decay estimates for wave equations with time-dependent coefficients, *J. Nonlinear Math. Phys.* 11 (4) (2004) 534–548.
- [3] F. Hirosawa, On the asymptotic behavior of the energy for the wave equations with time-depending coefficients, *Math. Ann.* 339 (4) (2007) 819–838.
- [4] F. Hirosawa, J. Wirth, C^m -theory of damped wave equations with stabilisation, *J. Math. Anal. Appl.* 343 (2) (2008) 1022–1035.
- [5] M. Reissig, K. Yagdjian, L_p - L_q decay estimates for the solutions of strictly hyperbolic equations of second order with increasing in time coefficients, *Math. Nachr.* 214 (2000) 71–104.
- [6] M. Reissig, On L_p - L_q estimates for solutions of a special weakly hyperbolic equation, Li, Ta-Tsien (ed.), *Proceedings of the conference on nonlinear evolution equations and infinite-dimensional dynamical systems*, Shanghai, China, June 12-16, 1995. Singapore: World Scientific. 153-164 (1997).
- [7] A. Galstian, L_p - L_q decay estimates for the equations with exponentially growing speed of propagation, *Appl. Anal.* 82 (3) (2003) 197–214.
- [8] J. Wirth, Solution representations for a wave equation with weak dissipation, *Math. Methods Appl. Sci.* 27 (1) (2004) 101–124.
- [9] M. Reissig, K. Yagdjian, One application of Floquet’s theory to L_p - L_q estimates for hyperbolic equations with very fast oscillations, *Math. Methods Appl. Sci.* 22 (11) (1999) 937–951.
- [10] S. Tarama, On the second order hyperbolic equations degenerating in the infinite order. — Example —, *Math. Jap.* 42 (3) (1995) 523–533.
- [11] W. Magnus, S. Winkler, *Hill’s equation.*, New York-London-Sydney: Interscience Publishers, a division of John Wiley & Sons. VIII, 127 p. , 1966.

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